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On The Transition Probabilities for the Fuzzy States of a Fuzzy Markov Chain

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ABSTRACT
In this paper the theory of fuzzy logic is mixed with the theory of Markov systems and the abstraction of a Markov system with fuzzy states introduced. The notions such as fuzzy transient, fuzzy recurrent etc., were introduced. The results based on these notions are introduced.
KEY WORDS: Fuzzy Markov chain, Fuzzy transition probability and Fuzzy random functions.

I. INTRODUCTION

Markov chains are a widely known statistical model of a number of real physical, natural, and social phenomenon. Examples comprise such as biology, medicine [5] economy and also problems related to speech recognition [4]. Fuzzy Markov chain is a robust system with respect to fuzzy transition probabilities which is not the case for the classical probabilistic Markov chains. Fuzzy Markov chains have an inherent application in Fuzzy Markov algorithms proposed by Zadeh[11]. The theory of Markov system offers an effective and powerful tool for describing the phenomena since numerous applied probability models can be adopted in their framework. Roughly speaking the Markov property strand in need of that knowledge of the current state of the system capable of furnishing all the information relevant to forecasting its future. A Markov system can be used to describe a phenomena that evolves over time according to probabilistic laws. Oliniok [6] has given in his article a more abstract and about a Markov fuzzy process with a transition possibility measure in an abstract state space. Bellman and Zadeh [1] were the first considered stochastic system in a fuzzy environment. By a fuzzy environment they mean the system has fuzzy goals and fuzzy constraints. A distinct approach combining fuzzy reasoning with Markov system can be found in [9] where fuzzy inference systems are used to estimate transition probabilities. Schweitzer[7] investigates the stationary distribution and fundamental matrix of an irreducible Markov chain as the transition probabilities vary slightly. Smith [8] describes the set of stationary probability vectors arising when the transition probabilities of an n-state Markov chain vary over a specified range.

Bhattacharyya [2] has introduced the Markov decision process with fuzzy states. Motivated by the fuzzy probability we introduce the concepts of fuzzy transient and related notions. We derive results based on these concepts.

In section 2 we briefly state some of the results related to dynamic fuzzy sets. These results will be frequently referred to in the subsequent sections.

In section 3 we introduce the preliminaries related to fuzzy markov chain and fuzzy transition probabilities.

II. DYNAMIC FUZZY SETS:

In what follows we denote the notions and notations of Dynamic fuzzy sets, introduced by Guangyuan wang et. At [3].

Let U be a non-empty usual set P(U) denote the set of all subsets in U and F(U) denote the set of all fuzzy subsets in U. For $A\in F(U)$ we define two subsets of U as follows.

$A_\alpha=\{x\in U;\ A(x)\geq\alpha\}$ for any $\alpha\in[0,1]$ (2.1)

$A_{\hat{\alpha}}=\{x\in U;\ A(x)>\alpha\}$ for any $\alpha\in[0,1]$ (2.2)

Where A(x) is the membership function of A.

DEFINITION:2.1

Let R be the real line and (R,B) be the Borel measurable space. Let $F_o(R)$ denote the set of fuzzy subsets $A:R\rightarrow[0,1]$ with the following properties

(1) $\{x\in R;\ A(x)=1\}\neq\phi$
(2) $A_\alpha=\{x\in R;\ A(x)\geq\alpha\}$ is a bounded closed interval in R for each $\alpha\in(0,1]$ ie
$A_\alpha=[A_\alpha^-, A_\alpha^+]$

Where $A_\alpha^-=\inf A_\alpha$ (2.3)

$A_\alpha^+=\sup A_\alpha$ (2.4)

$A\in F_0(R)$ is called bounded closed fuzzy number.

MARKOV PROCESS WITH FUZZY STATES

Let $X=\{x_1\ x_2,\ldots.x_n\}$ be a given set. A fuzzy pseudopartition or a fuzzy N-partition of X is a family of subset of α, denote by $A=\{A_1A_2\ldots.A_N\}$ with the corresponding membership function $\mu_1\mu_2\ldots.\mu_N$ which satisfies the orthogonality conditions $\sum_{r=1}^{N} \mu_{r(x_i)} = 1$ for all $x_i \in x$ and $0 < \sum_{r=1}^{N} \mu_{r(x_i)} < n$ for all $A_r \in A$ where n is a positive integer.

Let $F=\{F_1F_2\ldots F_N\}$ be the fuzzy state space, ie the set of fuzzy state for the problem. Fr is a fuzzy set on $S=\{1,2,\ldots,M\}$. Where S denotes the original non-fuzzy state space for the system, the element of which may or may not be exactly observable. Let $\mu_{Fr}(0): S \to [0,1]$ denote the membership function of the fuzzy state Fr, for r=1,2,…N. It is assumed that $\{F_1F_2\ldots F_N\}$ defines a pseudo partition of fuzzy set on S such that

$$\sum_{r=1}^{N} \mu_{Fr}(i) = 1 \text{ for all } i \in S \tag{2.5}$$

There are K possible alternatives available for each state. It is also assumed that the transition probability $P_{ij}(k)$ of the system moving from state i to state j under the alternative k is known for all $i,j \in S$ and for all k=1,2,…k. It is further assumed that the initial probability denoted by Q_{0i} of the system being in the non fuzzy state i is know, for each $i \in S$. As we shall be dealing with the problem on fuzzy state space Fr for all $Fr \in F$.

TRANSITION PROBABILITY FOR FUZZY STATES

Let X_t, X_t^f denote the non fuzzy and fuzzy state of the system at time t respectively. The Transition probability of the system moving from one fuzzy state to another can be computed as follows.

$$P_{Fr\,Fs} = \text{Prob}[X_1^f = F_s / X_0^f = F_r]$$

$$= \frac{\text{Prob}[X_1^f = F_s, X_0^f = F_r]}{\text{Prob}[X_0^f = F_r]} \tag{2.6}$$

Now using (2.6) we write

$$\text{Prob}[X_0^f = F_r] = \sum_{i=1}^{m} \mu_{Fr}(i)\, \Pr[X_0 = i]$$

$$= \sum_{i=1}^{m} \mu_{Fr}(i)\, Q_{0i} \tag{2.7}$$

And using (2.6) – (2.7) we get

$$\Pr[X_1^f = F_s, X_0^f = F_r]$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{m} \Pr[X_1 = j, X_0 = i]\, \mu_{FrFs}(i,j)$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{m} \Pr[X_1 = j, X_0 = i]\, \Pr[X_0 = i]\, \mu_{Fr}(i)\mu_{Fs}(j)$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{m} P_{ij}\, Q_{0i}\, \mu_{Fr}(i)\mu_{Fs}(j) \tag{2.8}$$

$$P_{Fr\,Fs} = \frac{\sum_{i=1}^{m}\sum_{j=1}^{m} P_{ij}\, Q_{0i}\, \mu_{Fr}(i)\mu_{Fs}(j)}{\sum_{i=1}^{m} Q_{0i}\, \mu_{Fr}(i)} \tag{2.9}$$

The above formula reduces to the following when all Q_{0i} are equal.

$$P_{Fr\,Fs} = \frac{\sum_{i=1}^{m}\sum_{j=1}^{m} P_{ij}\, \mu_{Fr}(i)\, \mu_{Fs}(j)}{\sum_{i=1}^{m} \mu_{Fr}(i)} \tag{2.10}$$

Writing

$$\sum_{i=1}^{m} \mu_{Fr}(i) = m_{Fr}$$

Equation (2.10) further simplifies to

$$P_{Fr\,Fs} = (m_{Fr})^{-1} \sum_{i=1}^{m}\sum_{j=1}^{m} P_{ij}\, \mu_{Fr}(i)\, \mu_{Fs}(j) \tag{2.11}$$

The m-step transition probability for fuzzy states is given by

$$\Pr[X_{n+m}^f = F_s / X_n^f = F_r] = P_{Fr\,Fs}(m) \tag{2.12}$$

$P_{Fr\,Fs}(m)$ gives the probability that from the fuzzy state F_r at the nth trial fuzzy state k is reached at (m+n) th trial in m-steps.

THEOREM:2.1
CHAPMAN – KOLMOGOROV EQUATION
For Fuzzy states Fr and Fs

$$p_{FrFs}^{(m+n)} = \sum_{r} p_{FrFj}^{(n)}\, p_{FjFs}^{(m)} \tag{2.13}$$

Proof:

The one – step transition probabilities $p_{FrFs}^{(1)}$ are denoted by P_{FrFs}

$$P_{FrFs}^{(2)} = \text{Prob}[X_{n+2}^f = F_s / X_n^f = F_r] \tag{2.14}$$

The fuzzy state Fs can be reached from the fuzzy state Fr in two steps through some intermediate fuzzy state r. Consider the fixed value of r.

$$\Pr[X_{n+2}^f = F_s / X_n^f = F_r]$$

$$= \Pr[X_{n+2}^f = F_s / X_{n+1}^f = F_j, X_n^f = F_r]\, \Pr[X_{n+1}^f = F_j, X_n^f = F_r]$$

$$= \Pr[X_{n+2}^f = F_s / X_{n+1}^f = F_j]\, \Pr[X_{n+1}^f = F_j, X_n^f = F_r]$$

$$\Rightarrow P_{FjFs}\, P_{FrFj} = P_{FrFj}\, P_{FjFs}$$

Since these intermediate fuzzy state r can assume values r = 1,2,… we have,

$$P_{FrFs}^{(2)} = \Pr[X_{n+2}^f = F_s / X_n^f = F_r]$$

$$= \sum_r pr\,[X_{n+2}^f = F_s, X_{n+1}^f = j \mid X_n^f = F_r]$$

$$= \sum_r p_{FrFj}\, p_{FjFs} \tag{2.15}$$

By induction we have

$$P_{FrFs}^{(m+1)} = \Pr[X_{m+n+1}^f = F_s / X_n^f = F_r]$$

$$= \sum_r pr[X_{m+n+1}^f = F_s / X_{n+m}^f = F_j]\, \Pr[X_{n+m}^f = F_j / X_n^f = F_r]$$

$$= \sum_r p_{FrFj}\, p_{FjFs}^{(m)}$$

Similarly we get

$$P_{FrFs}^{(m+1)} = \sum_r p_{FrFj}\, p_{FjFs}^{(m)}$$

In general we have

$$P_{FrFs}^{(m+n)} = \sum_r p_{FrFj}^{(n)}\, p_{FjFs}^{(m)} \tag{2.16}$$

III. CLASS PROPERTY

A class of fuzzy states is a subset of the fuzzy state space such that every fuzzy state of the class Communicates with every other and there is no other fuzzy state outside the class which communicates with all other fuzzy states in the class. A property defined for all fuzzy states of a chain is a class property if it possession by one fuzzy state in a class implies its possession by all fuzzy states of the same class. One such property is the periodicity of a state.

DEFINITION:3.1

Fuzzy state Fr is a return state if $P_{FrFr}^{(n)}>0$ for some n≥1. The period d_i of a return to state I is defined as the greatest common divisor of all m such that $P_{FrFr}^{(m)}>0$, Thus

$$d_i = G.C.D\ \{m;|\ P_{FrFr}^{(m)}|>0\ d\in (0,1]\}$$

Fuzzy state i is said to be aperiodic if d_i =1 and periodic if d_i >1. Clearly state i is aperiodic if $P_{FrFr}\neq 0$. It can be shown that two distinctive fuzzy states belonging to the same class have the same period.

Every finite Markov chain contains at last one closed set, i.e., the set of all fuzzy states or the fuzzy state space. If the chain does not contain any other proper closed subset other than the fuzzy state space, then the chain is called irreducible; the transition probability matrix of irreducible class is an irreducible matrix. In an irreducible Markov chain every fuzzy state can be reached from every other fuzzy state.

CLASSIFICATION OF STATES:

Suppose that a system starts with the fuzzy state Fr. Let $f_{FrFs}^{(n)}$ be the probabilities that it reaches the state Fs for the first time at the nth step. Let $P_{FrFs}^{(n)}$ be the probability that it reaches state Fs after n transitions. Let T_k be the first passage time to fuzzy state Fs. i.e.,

$$T_k = \min\{n\geq 1, Xn = Fs\}$$

And { $f_{FrFs}^{(n)}$} be the distribution of T_k given that the chain starts at the fuzzy state Fr. A relation can be established between $f_{FrFs}^{(n)}$ and $P_{FrFs}^{(n)}$ as follows. The relation allows $f_{FrFs}^{(n)}$ to be expressed in terms of $P_{FrFs}^{(n)}$. We state the following theorem.

THEOREM:3.1

Whatever be the state Fr and Fs

$P_{FrFs}^{(n)} = \sum_{r=0}^{n} f_{FrFs}^{(r)}\ p_{FsFs}^{(n-r)}\ ;\ \ n\geq 1$

With $P_{FrFs}^{(0)} = 1$, $f_{FrFs}^{(0)}=0$, $f_{FrFs}^{(1)}= P_{FrFs}$

It can also be written as

$P_{FrFs}^{(n)} = \sum_{r=0}^{n} f_{FrFs}^{(r)}\ p_{FsFs}^{(n-r)} + f_{FrFs}^{(n)}\quad ;\ n>1$

FIRST PASSAGE TIME DISTRIBUTION

Let F_{FrFs} denote the probability that starting with the fuzzy state Fr the system will ever reach the fuzzy state Fs. Clearly

$$F_{FrFs} = \sum_{n=1}^{\infty} f_{FrFs}^{(n)} \tag{3.2}$$

We have $\sup_{n\geq 1} p_{FsFs}^{(n)} \leq F_{FrFs} \leq \sum_{m\geq 1} p_{FrFs}^{(m)}\ \ for\ all\ \ n\geq 1$ (3.3)

We need to consider two cases

$$F_{FrFs} = 1, and\ \ F_{FrFs} < 1$$

The mean (first passage) time from fuzzy state Fr to fuzzy state Fs is given by

$$\mu_{FrFs} = \sum_{n=1}^{\infty} n f_{FrFs}^{(n)} \tag{3.4}$$

More over

$$\mu_{FrFr} = \sum_{n=1}^{\infty} n f_{FrFr}^{(n)} \tag{3.5}$$

Is known as the mean recurrence time for the fuzzy state Fr.

It is appropriate to express

$$d_i = G.C.D\ \{m;\ P_{FrFr}^{(m)}>0\} = G.C.D[m;\ f_{FrFs}^{(n)} > 0]$$

LEMMA:

Let $\{f_n\}$ be a sequence of fuzzy functions such that $f_n\geq 0$ $\sum fn = 1$and t≥1 be the greatest common divisor of those n for which $f_n>0$.

Let $\{u_n\}$ be another sequence such that $u_0=1$, $u_n=\sum_{r=1}^{n} f_r u_{n-r}; (n\geq 1)$then

$$\lim_{n\to\infty}(u_{nt})_\alpha = \frac{t}{\mu}$$

Where $\mu=\sum_{n=1}^{\infty} n f_n$the limit being zero when μ=∞ and

$$\lim_{N\to\infty}(U_N) = 0$$

Whenever N is not divisible by t.

DEFINITION:3.2

A fuzzy state Fr is said to be fuzzy persistent If F_{FrFr} =1

Ie., if $(F_{FrFr}) = \bigcup_{\alpha\in(0,1]} \alpha(F_{FrFr})_\alpha = 1$

Ie., if

$$\bigcup_{\alpha\in(0,1]} \alpha[(F_{FrFr})_\alpha^-, (F_{FrFr})_\alpha^+] = 1$$

Ie

$$\bigcup_{\alpha\in(0,1]} \alpha[(F_\alpha)_{FrFr}^-, (F_\alpha)_{FrFr}^+] = 1$$

DEFINITION:3.3

A fuzzy state Fr is said to be fuzzy transient if $F_{FrFr}<1$

Ie., if $(F_{FrFr}) = \bigcup_{\alpha\in(0,1]} \alpha(F_{FrFr})_\alpha < 1$

Ie., if

$$\bigcup_{\alpha\in(0,1]} \alpha[(F_{FrFr})_\alpha^-, (F_{FrFr})_\alpha^+] < 1$$

Ie

$$\bigcup_{\alpha\in(0,1]} \alpha[(F_\alpha)_{FrFr}^-, (F_\alpha)_{FrFr}^+] < 1$$

DEFINITION:3.4

A fuzzy persistent state Fr is said to be null persistent if $\mu_{FrFr}=\infty$ and said to be non-null persistent if $\mu_{FrFr}<\infty$.

Fuzzy state Fr is null persistent

If $(\mu_{FrFr}) = \bigcup_{\alpha\in(0,1]} \alpha(\mu_{FrFr})_\alpha = \infty$

Ie., if

$$\bigcup_{\alpha\in(0,1]} \alpha[(\mu_{FrFr})_\alpha^-, (\mu_{FrFr})_\alpha^+] = \infty$$

Ie

$$\bigcup_{\alpha\in(0,1]} \alpha[(\mu_\alpha)^{-}_{FrFr}, (\mu_\alpha)^{+}_{FrFr}] = \infty$$

If is non-null persistent if

$$\bigcup_{\alpha\in(0,1]} [\alpha(\mu_\alpha)^{-}_{FrFr}, (\mu_\alpha)^{+}_{FrFr}] < \infty$$

THEOREM 3.2

Fuzzy state Fr is fuzzy persistent if and only if

$$\sum_{n=0}^{\infty} p^{(n)}_{FrFs} = \infty \quad (3.6)$$

Proof:

Let

$$p^{(n)}_{FrFr}(s) = \sum_{n=0}^{\infty} p^{(n)}_{FrFr}\ s^n = 1 + \sum_{n=1}^{\infty} p^{(n)}_{FrFr}\ s^n$$

And

$$F^{(n)}_{FrFr}(s) = \sum_{n=0}^{\infty} f^{(n)}_{FrFr}\ s^n = \sum_{n=1}^{\infty} f^{(n)}_{FrFr}\ s^n$$

Be the generating function of the sequences $\{P_{FrFr}^{(n)}\}$and $\{f_{FrFr}^{(n)}\}$ respectively

We have from (3.6)

$$\sum_{n=0}^{\infty} p^{(n)}_{FrFs} = \infty p^{(n)}_{FrFr} = \sum_{r=0}^{\infty} f^{(n)}_{FrFr}\ p^{(n-r)}_{FrFr}$$

$$\bigcup_{\alpha\in(0,1]} \alpha(p^{(n)}_{FrFr})_\alpha$$

$$= \sum_{r=0}^{\infty} \bigcup_{\alpha\in(0,1]} \alpha(f^{(r)}_{FrFr})_\alpha \alpha(p^{(n-r)}_{FrFr})_\alpha \quad (3.7)$$

Multiplying both sides of (3.7) by s^n and adding for all n≥1 we get

$$\bigcup_{\alpha\in(0,1]} \alpha(p^{(s)}_{FrFr}) - 1 = \bigcup_{\alpha\in(0,1]} \alpha(F_{FrFr})_\alpha\ \alpha(p_{FrFr})_\alpha$$

Thus we have

$$\bigcup_{\alpha\in(0,1]} \alpha(p^{(s)}_{FrFr})_\alpha$$

$$= \frac{1}{1 - \bigcup_{\alpha\in(0,1]} \alpha(F_{FrFr})_\alpha} \quad (3.8)$$

Assume that Fr is fuzzy persistent which implies that

$$\bigcup_{\alpha\in(0,1]} \alpha(F_{FrFr})_\alpha = 1$$

Using Abels lemma

$$\lim_{s\to 1} \bigcup_{\alpha\in(0,1]} \alpha(F_{FrFr}{}^{(s)})_\alpha = 1$$

$$\lim_{s\to 1} \bigcup_{\alpha\in(0,1]} \alpha(P_{FrFr}{}^{(s)})_\alpha \to \infty$$

Since the coefficients of $(P_{FrFr})_\alpha$ are non-negative. We get

$$\sum \bigcup_{\alpha\in(0,1]} \alpha(P_{FrFr}{}^{(s)})_\alpha = \infty$$

Conversely if the state j is fuzzy transient then by Abels lemma

$$\lim_{s\to 1} \bigcup_{\alpha\in(0,1]} \alpha(F_{FrFr}{}^{(s)})_\alpha < 1$$

From (3.8)

$$\lim_{s\to 1} \bigcup_{\alpha\in(0,1]} \alpha(P_{FrFr}{}^{(s)})_\alpha < \infty$$

Since the coefficients

$$\sum_{n=0}^{\infty} \bigcup_{\alpha\in(0,1]} \alpha(P_{FrFr}{}^{(s)})_\alpha < \infty$$

$$\sum P_{FrFr} < \infty$$

THEOREM:3.3

If state j is fuzzy persistent non-null then as n→∞

(i) $P_{FrFr}{}^{(nt)} \to \frac{t}{\mu_{FrFr}}$ when fuzzy state Fr is periodic with period t and

(ii) $P_{FrFr}{}^{(n)} \to \frac{1}{\mu_{FrFr}}$ when fuzzy state Fr is aperiodic.

In case fuzzy state is persistent null then $P_{FrFr}{}^{(n)} \to 0$ as n→∞.

Proof:

Let fuzzy state Fr be persistent then

$\mu_{FrFr} = \sum_n n f^{(n)}_{FrFr}$ is defined

We have $P_{FrFs}{}^{(n)} = \sum_{r=0}^{n} f^{(r)}_{FrFr}\ p^{(n-r)}_{FrFr}$

We put $f^{(n)}_{FrFr}$ for f_n, $p^{(n)}_{FrFr}$ for u_n and μ_{FrFr} for μ in lemma 3.1

Applying the lemma we get

$P_{FrFr}{}^{(nt)} \to \frac{t}{\mu_{FrFr}}$ as n→∞

When state Fr is periodic with period t when fuzzy state Fr is aperiodic then

$P_{FrFr}{}^{(n)} \to \frac{1}{\mu_{FrFr}}$ as n→∞

In case fuzzy state Fr is persistent null $\mu_{FrFr} = \infty$ and $P_{FrFr}{}^{(n)} \to 0$ as n→∞

$$\mu_{FrFr} = \bigcup_{\alpha\in(0,1]} \alpha(\mu_{FrFr})_\alpha$$

$$= \bigcup_{\alpha\in(0,1]} \alpha[(\mu_{FrFr})^{-}_{\alpha}\ (\mu_{FrFr})^{+}_{\alpha}]$$

$$(\mu_{FrFr})_\alpha = (\mu_\alpha)_{FrFr} = \bigcap_{n=1}^{\infty} [(F_{\alpha n})^{-}_{FrFr}, (F_{\alpha n})^{+}_{FrFr}]$$

Where for any $\alpha \in (0,1]$ $\alpha_n = (1 - \frac{1}{n+1})^\alpha$

$(\mu_{FrFr})^{-}_{\alpha} = \inf(\mu_\alpha)_{FrFr}$

$= \inf [x \in R; \mu_{FrFr}(x) \geq \alpha]$

$(\mu_{FrFr})^{+}_{\alpha} = \mathrm{Sup}\ (\mu_\alpha)_{FrFr}$

$= \mathrm{Sup}\ [x \in R; \mu_{FrFr}(x) \geq \alpha]$

$$(\mu_{FrFr})_\alpha = \sum_n n(f^{(n)}_{FrFr})_\alpha$$

$$= \sum_n n(f^{(n)}_\alpha)_{FrFr}$$

We put $(f_\alpha^{(n)})_{FrFr}\ for\ fn, (p_\alpha^{(n)})_{FrFr}$ for u_n and $(\mu_\alpha)_{FrFr}$ for μ in Lemma (3.1) above. Applying Lemma we get,

$(p_\alpha^{(nt)})_{FrFr} \rightarrow \frac{t}{(\mu_\alpha)_{FrFr}}$ as n→∞.

When state j is periodic with period t.

$\bigcup_{\alpha\in(0,1]} \alpha (p_\alpha^{(nt)})_{FrFr} \rightarrow \frac{t}{\bigcup_{\alpha\in(0,1]} \alpha (\mu_\alpha)_{FrFr}}$

THEOREM:3.4

If fuzzy state Fs is fuzzy persistent null, then for every Fr

$$\lim_{n\to\infty} p_{FrFs}^{(n)} \rightarrow 0 \qquad (3.9)$$

And if fuzzy state Fs is aperiodic fuzzy persistent non-null then

$$\lim_{n\to\infty} p_{FrFs}^{(n)} \rightarrow \frac{F_{FrFs}}{\mu_{FsFs}} \qquad (3.10)$$

Proof:

$P_{FrFs}^{(n)} = \sum_{r=1}^{n} f_{FrFs}^{(r)}\ p_{FsFs}^{(n-r)}$

ie.,

$$\bigcup_{\alpha\in(0,1]} \alpha (P_{FrFs}{}^{(n)})_\alpha = \sum_{r=1}^{n} \bigcup_{\alpha\in(0,1]} \alpha (f_{FrFs}^{(r)})_\alpha \alpha (p_{FsFs}^{(n-r)})_\alpha$$

let n>m then

$$(P_{FrFs}{}^{(n)})_\alpha = \sum_{r=1}^{m} (f_{FrFs}^{(r)})_\alpha (p_{FsFs}^{(n-r)})_\alpha + \sum_{r=m+1}^{n} (f_{FrFs}^{(r)})_\alpha (p_{FsFs}^{(n-r)})_\alpha$$

$$\leq \sum_{r=1}^{m} (f_{FrFs}^{(r)})_\alpha (p_{FsFs}^{(n-r)})_\alpha + \sum_{r=m+1}^{n} (f_{FrFs}^{(r)})_\alpha (p_{FsFs}^{(n-r)})_\alpha$$

Since fuzzy state Fs is fuzzy persistent null$(p_{FsFs}^{(n-r)})_\alpha \rightarrow 0\ as\ n\rightarrow\infty$ for each α.

Further since

$$\sum_{m=1}^{\infty} (f_{FrFs}^{(m)})_\alpha < \infty$$

$$\sum_{r=m+1}^{n} (f_{FrFs}^{(r)})_\alpha \rightarrow 0\ for\ each\ \alpha \in (0,1]\ and\ n,m \rightarrow \infty$$

Therefore as n→∞

$$p_{FrFs}^{(n)} \rightarrow 0$$

$$\bigcup_{\alpha\in(0,1]} \alpha (P_{FrFs}{}^{(n)}) \rightarrow 0$$

Ie.,

$$p_{FrFs} \rightarrow 0$$

From the equation

$$(P_{FrFs}{}^{(n)})_\alpha - \sum_{r=1}^{m} (f_{FrFs}^{(r)})_\alpha (p_{FsFs}^{(n-r)})_\alpha \leq \sum_{r=m+1}^{n} (f_{FrFs}^{(r)})_\alpha \qquad (3.11)$$

Since fuzzy state is aperiodic, fuzzy persistent and non-null then by theorem

$$(p_{FsFs}^{(n-r)})_\alpha \rightarrow \frac{1}{(\mu_{FrFr})_\alpha}\ as\ n \rightarrow \infty$$

From the equation(3.11) we get n,m→ ∞

$$(p_{FsFs}^{(n-r)})_\alpha \rightarrow \frac{(F_{FrFs})_\alpha}{(\mu_{FrFr})_\alpha}$$

$$\bigcup_{\alpha\in(0,1]} \alpha (P_{FrFs}{}^{(n)})_\alpha \rightarrow \frac{\bigcup_{\alpha\in(0,1]} \alpha (F_{FrFs})_\alpha}{\bigcup_{\alpha\in(0,1]} \alpha (\mu_{FrFr})_\alpha}$$

Therefore,

$$p_{FsFs}^{(n-r)} \rightarrow \frac{F_{FrFs}}{\mu_{FrFr}}$$

THEOREM: 3.5

In an irreducible chain all the states are of the same type. They are either all fuzzy transient all fuzzy persistent null or all fuzzy persistent non-null. All the fuzzy state are aperiodic and in the latter case they all have the same period.

Proof:

Since the chain is irreducible, every fuzzy state can be reached from every other state. If Fr, Fs are any two states then Fr can be reached from Fs and Fs from Fr.

Ie., $p_{FrFs}^{(N)} = a > 0\ for\ some\ N \geq 1$

And $p_{FsFr}^{(M)} = b > 0\ for\ some\ M \geq 1$

Ie., $\bigcup_{\alpha\in(0,1]} \alpha (P_{FrFs}{}^{(N)})_\alpha = a > 0\ for\ some\ N \geq 1$

And $\bigcup_{\alpha\in(0,1]} \alpha (p_{FsFr}^{(M)})_\alpha = b > 0\ for\ some\ M \geq 1$

$$\bigcup_{\alpha\in(0,1]} \alpha (P_{FrFs}{}^{(n+m)})_\alpha = \bigcup_{\alpha\in(0,1]} \alpha (P_{FrFs}{}^{(m+n)})_\alpha$$

$$= \sum_{Fk} \alpha (p_{FrFk}^{(m)})_\alpha\ \ \alpha (P_{FkFs}{}^{(n)})_\alpha$$

$$(P_{FrFs}{}^{(n+m)})_\alpha = \sum_{Fk} \alpha (p_{FrFk}^{(m)})_\alpha\ \ (P_{FkFs}{}^{(n)})_\alpha$$

$$\geq (p_{FrFk}^{(m)})_\alpha (P_{FkFs}{}^{(n)})_\alpha\ \ for\ each\ Fk$$

Hence

$$(P_{FrFr}{}^{(n+N+M)})_\alpha \geq (P_{FrFs}{}^{(N)})_\alpha\ (P_{FsFs}{}^{(n)})_\alpha\ (P_{FsFr}{}^{(M)})_\alpha$$

$$\bigcup_{\alpha\in(0,1]} \alpha (P_{FrFr}{}^{(n+N+M)})_\alpha \geq \bigcup_{\alpha\in(0,1]} \alpha (P_{FrFs}{}^{(N)})_\alpha \bigcup_{\alpha\in(0,1]} \alpha (P_{FsFs}{}^{(n)})_\alpha \bigcup_{\alpha\in(0,1]} \alpha (P_{FsFr}{}^{(M)})_\alpha$$

$$(P_{FrFr})^{(n+N+M)} \geq (P_{FrFs}{}^{(N)})\ (P_{FsFs}{}^{(n)})\ (P_{FsFr}{}^{(M)}) = abP_{FrFr}{}^{(n)} \qquad (3.12)$$

$$and\ (P_{FsFs})^{(n+N+M)} \geq (P_{FsFr}{}^{(N)})\ (P_{FrFr}{}^{(n)})\ (P_{FrFr}{}^{(M)}) = abP_{FrFr}{}^{(n)} \qquad (3.13)$$

From the above it is clear that $\sum_n p_{FrFr}^{(n)}\ \ and \sum_n p_{FsFs}^{(n)}$ converge or diverge together. Thus the two fuzzy states Fr, Fs are either both fuzzy transient or fuzzy persistent.

Suppose that Fr is fuzzy persistent null then $(P_{FrFr}{}^{(n)})_\alpha \to 0 \; as \; n \to \infty$ for each $\alpha \in (0,1]$ from (3.12) $(P_{FsFs}{}^{(n)})_\alpha \to 0 \; as \; n \to \infty$ for each $\in (0,1]$. So that Fr is also fuzzy persistent null.

Suppose that Fr is persistent non-null and has period t then $(P_{FrFr}{}^{(n)})_\alpha > 0$ for each $\alpha \in (0,1]$ whenever n is a multiple of t.

Now

$$(P_{FrFr})^{(N+M)} \geq (P_{FrFs}{}^{(N)})_\alpha \, (P_{FsFs}{}^{(M)})_\alpha$$

$$\bigcup_{\alpha \in (0,1]} \alpha (P_{FrFr}{}^{(N+M)})_\alpha$$

$$\geq \bigcup_{\alpha \in (0,1]} \alpha (P_{FrFs}{}^{(N)})_\alpha \quad \bigcup_{\alpha \in (0,1]} \alpha (P_{FsFr}{}^{(M)})_\alpha$$

Ie.,

$$(P_{FrFr})^{(N+M)} \geq (P_{FrFs}{}^{(N)}) \, (P_{FsFs}{}^{(M)}) = ab > 0$$

So that (N+M) is a multiple of t . From equations (3.12 & 3.13),

$$(P_{FsFs})^{(n+N+M)} \geq ab (P_{FrFr})^{(n)} > 0$$

Thus (n+N+M) is a multiple of t and so t is the period of Fs also.

Theorem:3.6

An irreducible aperiodic Markov chain belongs to one of the following two classes.

(i) Either the states are all fuzzy transient or all nude fuzzy recurrent. In this case $p_{FrFs}{}^{(n)} \to 0$ and $n \to \infty$ for all fuzzy states Fr and Fs and there exists no stationary distribution.

(ii) Or else all states are positive fuzzy recurrent that is

$$\prod_{Fs} = \lim_{n \to \infty} p_{FrFs}^{(n)} > 0$$

In this case $\{\Pi_{Fs}$, Fs = 0,1,2,…} is a stationary distribution and there exists no other stationary distribution.

Proof:

We will first prove (ii)

$$p_{FrFs}^{(n)} = \bigcup_{\alpha \in (0,1]} \alpha (p_{FrFs}^{n})_\alpha$$

$$= \bigcup_{\alpha \in (0,1]} \alpha [(p_{FrFs}^{n})^{-}, (p_{FrFs}^{n})^{+}]$$

$$\sum_{Fs=1}^{M} p_{FrFs}^{(n)} \leq \sum_{Fs=0}^{\infty} p_{FrFs}^{(n)} = 1$$

Let $n \to \infty$ yields

$$\lim_{n \to \infty} \sum_{Fs=1}^{M} p_{FrFs}^{(n)} \leq 1$$

$$\sum_{Fs=1}^{M} \lim_{n \to \infty} (p_{FrFs}^{(n)}) \leq 1$$

$$\sum_{Fs=0}^{M} \prod_{Fs} \leq 1 \qquad for \; all \; M.$$

Which implies that

$$\sum_{Fs=0}^{\infty} \prod_{Fs} \leq 1$$

Now

$$(P_{FrFs}^{(n+1)})_\alpha = \sum_{Fk=0}^{\infty} (p_{FrFk}^{(n)})_\alpha (p_{FkFs})_\alpha$$

$$\geq \sum_{Fk=0}^{M} (p_{FrFk}^{(n)})_\alpha \, (p_{FkFs}{}^{k})_\alpha \qquad for \; all \; M.$$

Letting $n \to \infty$ yields

$$(\prod{}_{Fs})_\alpha \geq \sum_{Fk=0}^{m} (\prod{}_{Fk})_\alpha \; (p_{FkFs})_\alpha \;\; for \; all \; M.$$

Implying that

$$(\prod{}_{Fs})_\alpha \geq \sum_{k=0}^{\infty} (\prod{}_{Fk})_\alpha \, (p_{FkFs})_\alpha; \qquad Fs \geq 0$$

To show that the above is actually an actually equality, suppose that the inequality is strict for some Fs. Then upon adding these inequalities we obtain

$$\sum_{Fs=0}^{\infty} (\prod{}_{Fs})_\alpha > \sum_{Fs=0}^{\infty} \sum_{Fk=0}^{\infty} (\prod{}_{Fk})_\alpha \, (p_{FkFs})_\alpha$$

$$= \sum_{Fk=0}^{\infty} (\prod{}_{Fk})_\alpha \sum_{Fk=0}^{\infty} (p_{FkFs})_\alpha$$

$$= \sum_{Fk=0}^{\infty} (\prod{}_{Fk})_\alpha$$

Which is a contradiction. Fk=0

Therefore

$$(\Pi_{Fs})_\alpha = \sum_{Fk=0}^{\infty} (\Pi_{Fk})_\alpha (p_{FkFs})_\alpha \quad Fs = 0,1, ...$$

Putting $(p_{Fs})_\alpha = \frac{(\Pi_{\;Fs})_\alpha}{\sum_{Fk=0}^{\infty} \Pi_{\;Fk}}$

We observe that $\{p_{Fs}\}$ is a stationary distribution and hence at least one stationary distribution exists.

Now let $\{p_{Fs}, Fs = 0,1,2 \ldots\}$ be any stationary distribution

Then

$$p_{Fs} = p[X_n = Fs]$$

$$= \sum_{Fr}^{\infty} \Pr[X_n = Fs | X_0 = Fr] \, pr \, [X_0 = Fr]$$

$$= \sum_{Fr=0}^{\infty} (p_{FrFs}^{n})_\alpha (p_{Fr})_\alpha \qquad (3.14)$$

From (3.14) we note that

$$(p_{Fs})_\alpha \geq \sum_{Fr=0}^{M} (p_{FrFs}^n)_\alpha \, (p_{Fr})_\alpha \qquad for\ all\ M.$$

Letting n and then M approaches ∞ $yields$

$$(p_{Fr})_\alpha \geq \sum_{Fr=0}^{\infty} (\Pi_{Fs})_\alpha (p_{Fr})_\alpha = (\Pi_{Fs})_\alpha$$

To get the otherway and show that $(p_{Fr})_\alpha \leq (\Pi_{Fs})_\alpha$ use (3.14) and the fact that $(p_{FrFs}^n)_\alpha \leq 1$ to obtain

$$(p_{Fs})_\alpha \leq \sum_{Fr=0}^{M} (p_{FrFs}^n)_\alpha (\, p_{Fr})_\alpha + \sum_{Fr=M+1}^{\infty} (p_{Fr})_\alpha \;\; for\ all\ M$$

And letting n$\rightarrow \infty$ gives

$$(p_{Fs})_\alpha \leq \sum_{Fr=0}^{\infty} (\Pi_{Fs})_\alpha (\, p_{Fr})_\alpha + \sum_{Fr=m+1}^{\infty} (\, p_{Fr})_\alpha$$

Since $\sum_{Fr=0}^{\infty} (\, p_{Fr})_\alpha = 1$ we obtain
Upon letting M$\rightarrow \infty$ that

$$(p_{Fs})_\alpha \leq \sum_{Fr=0}^{\infty} (\Pi_{Fs})_\alpha (\, p_{Fr})_\alpha = (\Pi_{Fs})_\alpha \qquad (3.14)$$

If the states are fuzzy transient or null fuzzy recurrent and $\{P_{Fs}$, Fs=0,1,2….} is a stationary distribution then $(P_{FrFs})_\alpha \rightarrow 0$ which is clearly impossible. Thus for case(i) no stationary distribution exists and completes the proof.

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